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# Critical Lengths of Error Events in Convolutional Codes

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*Abstract* - If the calculation of the critical length is based on the expurgated exponent, the length becomes nonzero for low error probabilities. This result applies to typical long codes, but it may also be useful for modeling error events in specific codes.

## I. Introduction.

Forney [1] introduced the concept of the critical length for long convolutional codes as the length of the error events that dominate the lower bound on error probability. This analysis, which also appears in Viterbi and Omura's text [2], indicates that for rates below $R_0$ the critical length is 0. In the discussion of the derivation we make some comments on the relationship between distances and error exponents.

## II. Error exponents for block codes and convolutional codes.

For simplicity we shall discuss only the performance on binary symmetric channels. The error exponents of block codes and convolutional codes are discussed in [2] and many other books.

The easiest part of the error exponent decreases linearly with the rate of the code and reaches zero at $R_0$. This follows from the union bound applied to an ensemble of randomly chosen codes. For high rates the bound may be improved so that nonzero exponents are obtained for rates up to the channel capacity, but we shall not consider such rates here.

For low rates the error probability is dominated by the error events of low weight. A typical linear code has a weight distribution that is obtained by scaling the binomial distribution to the right number of codewords. The expurgated ensemble of codes is obtained by deleting codes with minimum distances below the Gilbert bound. The expurgated exponent expresses the probability of making a decision in favour of one of the few words at distance $d_{min}$ from the transmitted word.

For convolutional codes, the performance may be bounded by considering a sequence of block codes obtained by terminating the convolutional code at different rates [1]. The bounds obtained in this way are related to the block code bounds by the inverse concatenation construction: A tangent to the block code bound at the point corresponding to rate R intersects the rate axis in (r,0) where r is the rate of the convolutional code. The intersection with the exponent/distance axis gives the performance bound for the convolutional code.

## III. The critical length.

For a given channel and a convolutional code of rate r, the inverse concatenation construction gives the rate, R, of the critical block code, and thus the number of information symbols in the critical error events. If the construction is based on the random coding exponent, the critical length equals 0 for all rates below $R_0$. However, this result is based on the existence of codewords of very low weight in some codes (on the average less than one). For typical codes, the expurgated exponent should be used, and the critical length becomes a positive function which increases with increasing rate. For $r=R_0$ there is still a discontinuous increase in the critical length due to the straight line portion of the error exponent.

This derivation of the critical length also relates the error events to the codewords of minimal weight. The codewords of weight $d_{free}$ are minimum weight codewords in the critical block code. Again the inverse concatenation construction applies, and the rate of the critical code is known to increase with the rate of the convolutional code. This property of the distances was clearly not in agreement with the result that the error probability should be dominated by error events of length 0.

In simulations of decoding systems it is sometimes assumed that the length of the error events approximately follows a geometric distribution. While the exponential decrease in probability is a very good approximation for bursts of length greater than the critical length, shorter bursts occur with smaller probabilities.

## IV. Performance as a function of the error probability.

While error exponents and related results are usually given as functions of the code rate in the information theory literature, it is customary in communications to fix the code and study the performance as a function of the signal to noise ratio. For the binary symmetric channel (hard decision), we may consider the performance of a code with fixed rate as a function of the error probability. For small error probabilities the critical length equals the dimension of the critical block code, and there is a discontinuous increase at the error probability which makes $R_0$ for the channel equal r.

## References.

1. G.D. Forney, "Convolutional codes II: Maximum-likelihood decoding", Information and Control, vol.25, pp. 222-250, 1974.

2. A.J. Viterbi and J.K. Omura, Principles of Digital Communication and Coding, New York, McGraw-Hill, 1979.

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